Assignment 6

Hand in no. 3, 7 and 8 by October 24.

1. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}$:
   (a) $[1,2) \cup (2,5) \cup \{10\}$.
   (b) $[0,1] \cap \mathbb{Q}$.
   (c) $\bigcup_{k=1}^{\infty} (1/(k+1), 1/k)$.
   (d) $\{1, 2, 3, \cdots\}$.

2. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}^2$:
   (a) $R \equiv [0,1) \times [2,3) \cup \{0\} \times (3,5)$.
   (b) $\{(x,y) : 1 < x^2 + y^2 \leq 9\}$.
   (c) $\mathbb{R}^2 \setminus \{(1,0), (1/2,0), (1/3,0), (1/4,0), \cdots\}$.

3. Describe the closure and interior of the following sets in $C[0,1]$:
   (a) $\{f : \ f(x) > -1, \ \forall x \in [0,1]\}$.
   (b) $\{f : \ f(0) = f(1)\}$.

4. Let A and B be subsets of (X,d). Show that
$$\overline{A \cup B} = \overline{A} \cup \overline{B} \ .$$
   Is it true that
$$\overline{A \cap B} = \overline{A} \cap \overline{B} \ ?$$

5. Show that $\overline{E} = \{x \in X : \ d(x,E) = 0\}$ for every non-empty $E \subset X$.

6. Let $E \subset (X,d)$. Show that E° is the largest open set contained in E in the sense that $G \subset E^\circ$ whenever $G \subset E$ is open.

7. Determine whether $\mathbb{Z}$ and $\mathbb{Q}$ are complete sets in $\mathbb{R}$.

8. Does the collection of all differentiable functions on $[a,b]$ form a complete set in $C[a,b]$?

9. Let (X,d) be a metric space and $C_b(X)$ the vector space of all bounded, continuous functions in X. Show that it forms a complete metric space under the sup-norm.

10. We define a metric on $\mathbb{N}$, the set of all natural numbers by setting
$$d(n,m) = \left| \frac{1}{n} - \frac{1}{m} \right| \ .$$
    (a) Show that it is not a complete metric.
    (b) Describe how to make it complete by adding one new point.

11. Optional. Let (X,d) be a metric space. Fixing a point $p \in X$, for each x define a function
$$f_x(z) = d(z,x) - d(z,p).$$

(a) Show that each f_x is a bounded, uniformly continuous function in X.

(b) Show that the map $x \mapsto f_x$ is an isometric embedding of (X, d) to $C_b(X)$. In other words,

$$\|f_x - f_y\|_\infty = d(x, y), \quad \forall x, y \in X.$$

(c) Deduce from (b) the completion theorem asserting that every metric space has a completion.

This approach is much shorter than the proof given in the appendix of Chapter 3 . However, it is not so inspiring.

12. Optional. Let $\mathcal{K}$ be the collection of all non-empty closed and bounded sets in $\mathbb{R}^n$. We introduce a metric called the Hausdorff metric on $\mathcal{K}$ as follows. The set E_ε is defined to be the set $\{x + \varepsilon z : \ x \in E, \ |z| < 1\}$, $\varepsilon > 0$. For closed and bounded E, F, define

$$\rho_H(E, F) = \inf \{\varepsilon : \ F \subset E_\varepsilon, \ E \subset F_\varepsilon\} .$$

(a) Show that

$$E_\varepsilon = \{y \in \mathbb{R}^n : \ d(y, E) < \varepsilon \} .$$

(b) Show that

$$\rho_H(E, F) = \max \left\{ \sup_{x \in E} d(x, F), \ \ \sup_{y \in F} d(y, E) \right\} ,$$

where $d(x, F)$ is the Euclidean distance from x to F.

(c) Show that ρ_H is a metric on $\mathcal{K}$.

(d) Let $\{K_n\}, K_{n+1} \subset K_n$, be a descending sequence in $\mathcal{K}$. Show that

$$\rho_H(K_n, K_\infty) \to 0, \qquad \text{as } n \to \infty ,$$

where $K_\infty = \cap_j K_j \neq \phi$.

13. Optional. In the previous problem, it is shown that the Hausdorff metric makes $\mathcal{K}$, the set of all non-empty closed and bounded sets in $\mathbb{R}^n$, a metric space. Now show that it is complete. Hint: Let $\{K_n\}$ be a Cauchy sequence in $\mathcal{K}$ and consider the descending family $H_n = \overline{\bigcup_{j \geq n} K_j}$. Apply Problem 12(c) and show $K_n \to \bigcap_{k \geq 1} H_k$.